Partial Differential Equations - Midterm exam

You have 2 hours to complete this exam. Please show all work. Each question is worth 25 points for a total of 100 points. Be sure to quote clearly any theorems you use from the textbook or class. Good luck!

(1) Using $\frac{1}{\sqrt{2}}, \{\sin(kx), \cos(kx)\}_{k\in\mathbb{N}}$ as your basis for $L^2([-\pi, \pi])$
a) State and prove Bessel's inequality.
b) State and prove Parseval's inequality. You may use the fact the basis is complete without proof.

(2) The telegrapher's equation $u_{tt} + au_t = c^2 u_{xx}$ with $a > 0$ models the vibration of a string under frictional damping.
a) Assume for solutions to this equation that there exists $\alpha > 1/2$ and $C(t) > 0$ such that $|u_t(t,x)|, |u_x(t,x)| \leq \frac{C(t)}{|x|^\alpha}$ for each fixed t and all sufficiently large $|x| \gg 0$. Show that under these assumptions the wave energy is a non increasing function of t. Recall that the wave energy is given by

$$E(t) = \int_{-\infty}^{\infty} \frac{1}{2}\left(u_t^2 + c^2 u_x^2\right) dx. \tag{0.1}$$

b) Prove uniqueness of such solutions to the initial value problem for the telegrapher's equation.

(3) a) Find a solution to the initial value problem

$$u_t + u_x = 0 \quad u(1,x) = \frac{x}{1+x^2}.$$

b) Solve the initial value problem

$$u_t + 2u_x = 1 \quad u(0,x) = e^{-x^2}.$$

(4) a) Let

$$v_n(x) = \frac{2nx}{1+n^2x^2}. \tag{0.2}$$

What does the sequence $v_n(x)$ converge to as $n \to \infty$? Does the sequence converge pointwise? Uniformly? Justify your answer.
b) Compute the Fourier series of $f(x) = x$ on the interval $[-\pi, \pi]$. Write down explicitly to what function the Fourier series converges, draw the function, and justify the convergence by stating the appropriate theorem.